Identification of clusters of companies in stock indices via Potts super-paramagnetic transitions

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The clustering of companies within a specific stock market index is studied by means of super-paramagnetic transitions of an appropriate q-state Potts model where the spins correspond to companies and the interactions are functions of the correlation coefficients determined from the time dependence of the companies' individual stock prices. The method is a generalization of the clustering algorithm by Domany *et. al.* to the case of anti-ferromagnetic interactions corresponding to anti-correlations. For the Dow Jones Industrial Average where no anti-correlations were observed in the investigated time period, the previous results obtained by different tools were well reproduced. For the Standard & Poor's 500, where anti-correlations occur, repulsion between stocks modify the cluster structure.

Stock market indices, like the Dow Jones (DJ) or Standard & Poor's 500 (S&P 500) are used as indicators of the status of the markets. They are averaged values of a different number of selected companies indicative of the economy of a given market. It is of both theoretical and practical importance to analyze how individual contributions to the average behave. The customary approach in the financial literature focuses on the investigation of the properties of the covariance matrix. Here we take a different approach aiming to identify the presence of a hierarchical structure inside the set of stocks simultaneously traded in a market. The identification of the hierarchy of clusters is of central importance both from the point of view of understanding the dynamics of the stock index and for portfolio optimization [1,2]. As far as we know this question was first analyzed by Mantegna by means of the minimal spanning tree method [3–5], see also [6]. Here we analyze the problem of clustering of companies in the S&P 500 and the DJ indices by a different method based on the q-state Potts model which turns out to be particularly suitable to handle anti-correlations.

The idea to use the super-paramagnetic (SPM) ordering of a q-state Potts model for cluster identification is due to Domany *et. al* [7–9]. They start from a set of points which lie in a metric space where the mutual distances of the points are known. By introducing a distance dependent ferromagnetic (FM) interaction between Potts spins assigned to the points at appropriately chosen temperatures the close points within a cluster feel strong interaction and align while far clusters point into different "Potts directions". The functional dependence of the interaction on the distance should be chosen in an appropriate way. For a given interaction the possible hierarchic clustering shows up in a series of SPM transitions.

We have generalized this method by dropping the condition of the metric and allowing negative (anti-ferromagnetic, AFM) couplings. The coupling between the pair of Potts spins (*i.e.* companies) is in our case the explicit function of the correlation coefficient and it is FM for positive correlations and AFM for anti-correlations (the latter are present in the S&P 500). This way we naturally take into account the "repulsion" between negatively correlated companies (and clusters of companies) – an important aspect for portfolio optimization. In order to estimate the effect of the anti-correlations we carried out calculations with only FM interactions (*i.e.* we took the absolute values of the correlation coefficients) and with correctly signed interactions too. We found that the difference – in our set of data – can be observed only at the ground state, *i.e.* for the main (dominant) cluster structure.

Consider a q state, inhomogeneous Potts model: $s_i = 1, ..., q$, where $i = 1, ..., N$. N is the number of points one should arrange in clusters (the number of companies in the considered situation). The cost function will be the Hamiltonian:

$$H = -\sum_{(i,j)} J_{ij}\delta_{s_i,s_j}. \tag{1}$$

The coupling J_{ij} is a function of the correlation coefficient c_{ij} between the time evolution of the logarithmic daily price return $Y_i = \log(P_i(t)) - \log(P_i(t-1))$ of the stock of companies i and j. $P_i(t)$ is the closure price of the stock i at the day t. The correlation coefficient can be computed as follows:

$$c_{ij} = \frac{\langle Y_i Y_j\rangle - \langle Y_i\rangle\langle Y_j\rangle}{\sqrt{\left(\langle Y_i^2\rangle - \langle Y_i\rangle^2\right)\left(\langle Y_j^2\rangle - \langle Y_j\rangle^2\right)}} \in [-1,1]. \tag{2}$$

Here $\langle ...\rangle$ is a temporal average performed on all the trading days of the investigated time period which ranges

from July 3rd, 1989 to October 27th, 1995.

The Potts model can be used for cluster identification in the following way. Let us first consider the simpler, FM case [1], *i.e.*, $J_{ij} \geq 0$. These couplings are functions of the property the clustering should be based upon – in our case this is the correlation coefficient. If the set of spins are interrelated in a way that each pair of spins can be connected through a path via non-vanishing J_{ij}-s the ground state of the system is all spins pointing into one Potts-direction, *i.e.*, they build a single cluster. As the temperature is increased, weak bonds break easier than the strong ones and transition to a SPM phase takes place where clusters of spins have a specific Potts magnetization but the net magnetization of the whole system is zero. The clusters identified in this manner are those we have been looking for. Depending on the interactions, the system may go through a sequence of such transitions signalizing the hierarchical cluster structure. The transitions are best indicated by monitoring the peak structure in the susceptibility and the clusters are then identified by means of the spin-spin correlation functions.

The method is easily generalized to the case where repulsion between pairs of points is present, in our case, if there are anti-correlations between companies as it is the case *e.g.* for the S&P 500. This latter case is important when AFM interactions are also present, because of low temperature behaviour.

For the above reasons we make the following choice for the interaction:

$$J_{ij} = \mathrm{sgn}(c_{ij}) \left(1 - \exp \left\{ -\frac{n-1}{n} \left[\frac{c_{ij}}{a} \right]^n \right\} \right), \qquad (3)$$

where c_{ij} is the correlation coefficient between companies i and j. The parameters $(a,\, n)$ should be chosen so that the super-paramagnetic state exist, but inside this region the result will be not too sensible on the choice. The fine tuning serves to be able to observe the transitions more clearly, i.e. make peaks in the susceptibility function sharper, and the constant regions between them larger. A possible determination of parameter a is the average of the largest correlation coefficients for each spin: $a = 1/N \sum_{i=1}^{N} \max_j(c_{ij})$. The power n tunes the range of interaction, the factor $n/(n-1)$ in the exponent shifts the inflection point of the interaction function.

The order parameter is:

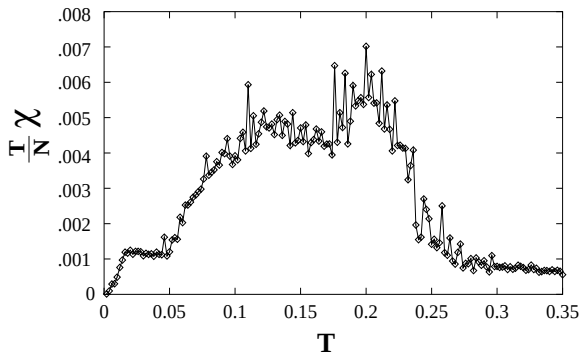
$$m = \frac{N_{max}/N - 1/q}{1 - 1/q}, \qquad (4)$$

where q is the number of states that a spin can have, N is the number of spins, and N_{max} is the maximal number of spins which are in the same state. The value of the parameter q is determined by a trial and error optimization: Too small q hinders the identification of the SPM clusters since different clusters are then forced to point into the same "Potts direction". Too large q makes the calculations more cumbersome. The results depend only weakly on the value of q.

The (first order-like) SPM transitions are identified by the peaks in the susceptibility $\chi = N(\langle m^2 \rangle - \langle m \rangle^2)/T$, which has to be measured as the function of the temperature. (For more convenience we studied the $\hat{\chi}(T) = T\chi(T)/N$ function.) In the simplest non-trivial case there are two transitions as temperature is increased: FM $\rightarrow$ SPM and SPM $\rightarrow$ PM. However, inside the range of the SPM phase one can often observe several peaks too, meaning that there are more than one characteristic cluster configurations. As the temperature increases the clusters break up into sub-clusters: A hierarchical structure is revealed.

The susceptibility $\hat{\chi}$, is approximately constant between two peaks. At this temperature regime the clusters are identified by means of the spin-spin correlation function: If the correlations between two spins exceed a given threshold (*e.g.* 0.5), the spins (companies) are considered to belong to the same cluster. The result is not sensitive on the choice of the threshold value. The distribution of the spin-spin correlation has two peaks, one of them is near to the value zero and the other near to the value one. The probability that the correlation of two spins lies between these two values is low.

First we analyzed the companies of the Dow Jones index which includes $N = 30$ companies. No negative correlation coefficient was found for this data set ($c_{ij} \geq 0$). In this purely FM case the simulation could be done with the efficient Swendsen-Wang method [10]. The parameters of the interaction (3) were set to $a = 0.43$, $n = 8$. The temperature dependence of the susceptibility $\hat{\chi}$, is shown in Fig. 1.



[1] It should be mentioned here that there are no anti-correlations in the DJ index in the investigated time period.

FIG. 1. Temperature dependence of $\hat{\chi} = T\chi/N$ in the FM case (Dow Jones companies). The parameters are: $a = 0.43$, $n = 8$ and $q = 100$.

We analyzed the spin-spin correlations at four different temperature levels. We got the following clusters (we use the standard notation for the companies):
$T = 0.03$
1) AA, ALD, AXP, BA, BS, CAT, CHV, DD, DIS, GE, GM, IP, JPM, KO, MCD, MMM, MO, MRK, PG, T, TX, UK, UTX, XON,
$T = 0.13$
1) AA, DD, GE, GM, IP, JPM, KO, MCD, MMM, MO, MRK, PG, T
2) CHV, TX, XON
$T = 0.15$
1) CHV, TX, XON
2) DD, GE, IP, KO, MCD, MMM, MO, MRK, PG, T
$T = 0.28$
1) CHV, TX, XON

Due to the small system size we have large fluctuations and the identification of clusters is not unambiguous. Nevertheless, the clusters do well match with the deterministic minimal spanning tree calculation by Mantegna though the hierarchical structure of the clusters is not so clearly represented.

The other system we analyzed were the companies of the S&P500 consisting of $N = 443$ companies. First we looked at the case of strictly FM interactions, *i.e.* we took the absolute value of the correlation coefficients. The Fig. 2 presents the $\hat{\chi}(T)$ function. The observed large fluctuations in the susceptibility function is related to the narrow distribution of correlation coefficients (Fig. 4). This makes the separation of clusters sensitive to the temperature.

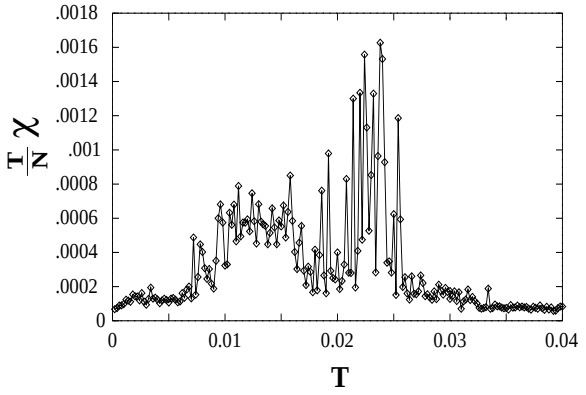


FIG. 2. Temperature dependence of $\hat{\chi} = T\chi/N$ in the FM case (S&P500 companies) with the parameters: $a = 0.65$, $n = 10$ and $q = 50$.

This system represents a clear hierarchical structure of the clusters, see Fig. 3. In the boxes the number of the elements of the cluster are indicated. The efficiency of the method is represented by the fact that the figure matches well with the corresponding result obtained by the minimal spanning tree method, including the specific composition of the clusters. *E.g.*, at the lowest level of the hierarchy (highest temperature in the SPM phase) the different industrial branches can be clearly identified: Oil, electricity, gold mining, etc. companies build separate clusters.

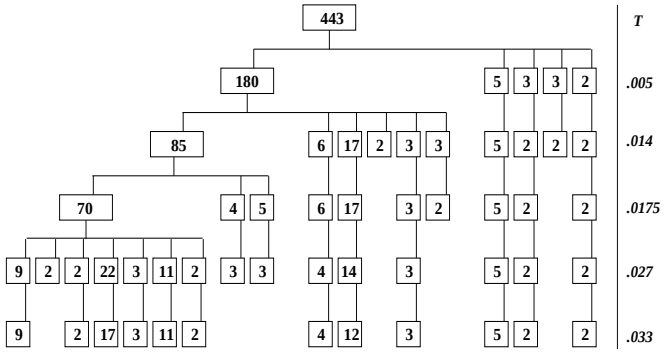


FIG. 3. The hierarchical structure of clusters of the S&P500 companies in the FM case. The clusters consisting of single companies are not indicated.

It is misleading in the above picture that we take strong anti-correlations equal to strong correlations: We have to take into account the anti-correlations properly. To simulate an inhomogeneous Potts model with long range and altering sign interactions seems to be a hard task, even the definition of the order parameter is non-trivial. Fortunately, in our case the negative interactions play a secondary role as compared to the ferromagnetic ones: They are much weaker and their number is much less. This is shown in Fig. 4 where the distribution of the correlation coefficients is presented and not the interactions.

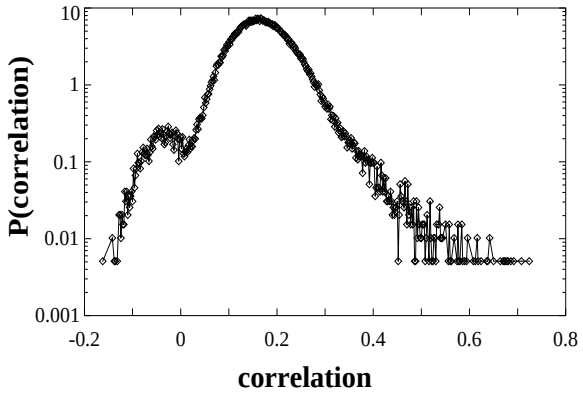


FIG. 4. Distribution of the measured correlation coefficients (S&P500 companies).

Further simplification is that almost all the negative interaction is associated with only five companies (gold mining) building one cluster in the FM version too since

there are quite strong positive correlations between them. The consequence is that except of very low temperatures there will be no significant difference to the FM case and the simulation of our system can be carried out by a simple Metropolis algorithm, there is no need for the application of more sophisticated tools like the multi-canonical algorithm.

However, the determination of the low temperature configurations is not straightforward. The system falls easily into a local minimum and it takes much simulation time to get out of there. Therefore we used a process in the spirit of the simulated tempering [11]. We excite the system to a higher temperature level. Then the temperature is lowered gradually so that at each temperature level we keep the configuration according to the minimal energy and keep the records of the best candidates of the low energy configurations.

Fig. 5 represents the configurations and their energy values we got for this temperature range.

configuration	number of companies	energy
C_1	438, 5	-672.37417
C_2	443	-672.35707

FIG. 5. Energies and configurations at low temperature in the AFM case of the S&P500 companies.

Clearly, the pure FM state (C_2) will not have the lowest energy value. This is not very surprising if one knows that those five companies have to fall into a separate cluster.

Our goal was to identify the clusters of companies at two stock indices, and to show that the repulsion between the companies - due to the negative correlation coefficients - can modify the cluster structure. Due to the distribution of the correlation coefficients in our system this effect is significant only at the ground state, *i.e.*, in the dominant cluster structure. Nevertheless, we think that our analysis demonstrates the importance of the repulsion effects in the clustering problem.

ACKNOWLEDGMENTS

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